Definition 4.17: $r(R)$, $s(R)$ and $t(R)$

Let R be a relation on a non-empty set A. The **reflexive (symmetric or transitive) closure** of R is a relation R' *on* A, ***such that*** R' satisfies the following conditions:

- R' is reflexive (symmetric or transitive).
- $R \subseteq R'$
- For any reflexive (symmetric or transitive) relation R'' **on A that contains** R**, we have** $R' \subseteq R''$ **.**

The reflexive closure of R is usually denoted by $r(R)$, the symmetric closure by $s(R)$, and the transitive closure by $t(R)$.

↳ Construction of the Transitive Closure of Relation *R*

- For a relation ***R*** on a non-empty set ***A***, the reflexive closure ***r*(*R*)**, symmetric closure ***s*(*R*)**, and transitive closure ***t*(*R*)** can be constructed.
- The ***reflexive closure R′*** of *R* is a relation obtained by adding all necessary pairs to ensure reflexivity, and it is the smallest superset. It can be defined as: $R' = R \cup \{(a,a) \mid a \in A\}$
- The ***symmetric closure R′*** of ***R*** is a relation obtained by adding all necessary pairs to ensure symmetry, and it is the smallest superset. It can be defined as: $R' = R \cup \{(b,a) \mid (a,b) \in R\}$

- The ***transitive closure R′*** of ***R*** is a relation obtained by adding all necessary pairs to ensure transitivity, and it is the smallest superset. It can be defined as:

For each pair of elements a,c∈A, if there exist one or more elements $b_1,b_2,\ldots,b_n$ such that $(a,b_1),(b_1,b_2),\ldots,(b_{n-1},b_n),(b_n,c)$ are all in ***R*** , then (a,c) should be in ***R′***.

↳ Closure Theorem of Relations

- **Theorem 4.7:** Closure Theorem of Relations.

  Let R be a relation on A, then we have:

  (1) $r(R)=R \cup R^0$

  (2) $s(R)=R \cup R^{-1}$

  (3) $t(R)=R \cup R^2 \cup R^3 \cup \ldots$

Explanation:

- For a finite set A (where $|A|=n$) , the union in (3) will have at most R^n.
- *If* R is reflexive, then $r(R)=R$; If R is symmetric, then $s(R)=R$; If R is transitive, then $t(R)=R$.

- **Proof of Theorem 4.7** (Proving (1)).
  - Proof of **(1)** $r(R)=R\cup R^0$, It is sufficient to show that $R\cup R^0$ satisfies the closure definition.
  - Proof that $R\cup R^0$ is a reflexive relation

    Since $R\cup R^0$ contains R , **and by** $I_A\subseteq R\cup R^0$, we can conclude that $R\cup R^0$ is reflexive on A.
  - Proof that $R\cup R^0$ is the smallest reflexive relation containing R .

    We need to show that no reflexive relation smaller than $R\cup R^0$ exists that contains R.

    Assume R' is a reflexive relation that contains R *and* and is smaller than $R\cup R^0$ $I_A\subseteq R'$, $R\subseteq R'$, Therefore, we have $R\cup R^0=I_A\cup R\subseteq R'$, which contradicts the assumption that R' is smaller than $R\cup R^0$.

- **Proof of (3) $t(R)=R\cup R^2\cup R^3\cup\ldots$**
  - Consider arbitrary pairs **$<x,y>$ and $<y,z>$**

    $$<x,y>\in R\cup R^2\cup R^3\cup\ldots \wedge <y,z>\in R\cup R^2\cup R^3\cup\ldots$$

    $$\Rightarrow <x,z>\in R\cup R^2\cup R^3\cup\ldots$$

    Therefore, by the transitivity of $R\cup R^2\cup R^3\cup\ldots$ **We have**

    $$t(R)\subseteq R\cup R^2\cup R^3\cup\ldots$$

  - Next, we prove by induction that $R^n\subseteq t(R)$.

    For $n=1$, the statement is obviously true. Assume it holds for $n=k$.

    For any $<x,y>$, **we have**

    $$<x,y>\in R^{k+1}\Rightarrow <x,y>\in R^k\circ R \Rightarrow \exists t\,(<x,t>\in R^k \wedge <t,y>\in R)$$

    $$\Rightarrow \exists t\,(<x,t>\in t(R)\wedge <t,y>\in t(R)) \Rightarrow <x,y>\in t(R) \quad (t(R)\text{ transitive})$$

    Thus, $R\cup R^2\cup R^3\cup\ldots\subseteq t(R)$

- Let the relation matrices of R, $r(R)$, $s(R)$, $t(R)$ **be** M, M_r, M_s **and** M_t , respectively. Then, we have:

$$M_r = M + E$$

$$M_s = M + M'$$

$$M_t = M + M^2 + M^3 + \ldots$$

- where E is the identity matrix of the same order as M, and M' is the transpose of M.

Note: In the above equations, the matrix elements are added using logical addition.

↳ Closure Operations on Relation Graphs

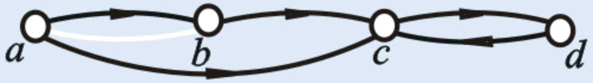
- Let the relation graphs of R, $r(R)$, $s(R)$, $t(R)$ be denoted by G, G_r, G_s, G_t , respectively.

  Then, the vertex sets of G_r, G_s, G_t are the same as the vertex set of G.
- In addition to the edges of G , new edges are added in the following ways：
  - For each vertex in G, if there is no cycle, add a cycle. The resulting graph is G_r.
  - For each directed edge $x_i \to x_j$, (with $i \neq j$), add a reverse edge $x_j \to x_i$. The resulting graph is G_s.
  - For each vertex x_i in G, examine all paths starting from x_i ， if there is no edge from x_i to any node x_j in the path, add the corresponding edge. After checking all vertices, the resulting graph is G_t .

4.3.2 Closure of Relations

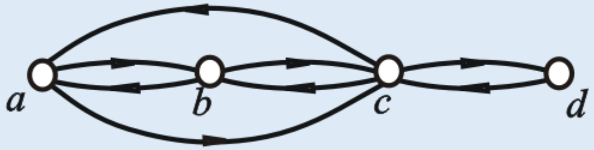
↳ Closure Operations on Relation Graphs (e.g.)

e.g.≫ **Example:** Let $A=\{a,b,c,d\}$, $R=\{<a,b>,<a,c>,<b,c>,<c,d>,<d,c>\}$, R and $r(R)$, $s(R)$, $t(R)$ the relation graph is shown.

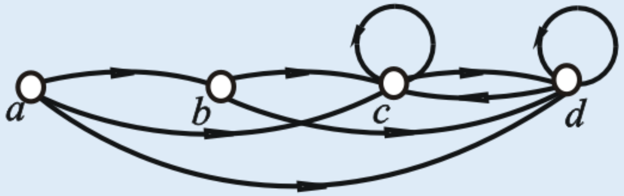


R

$r(R)$



$s(R)$



$t(R)$

- **Algorithm Idea:** Consider a sequence of matrices M_0, M_1, …, M_n of size n+1, where the element in the i-th row and j-th column of matrix M_k is denoted as $M_k[i,j]$. For k=0,1,…,n, $M_k[i,j]=1$ if and only if there exists a path from x_i to x_j in the relation graph of R , and this path passes through only the vertices in $\{x_1, x_2, …, x_k\}$ except for the endpoints. It is easy to prove that M_0 is the relation matrix of R , and M_n corresponds to the transitive closure of R .
- **Warshall Algorithm:** Starting from M_0, calculate M_1, M_2, …, until M_n .

  From $M_k [i, j]$ *to compute* $M_{k+1}[i, j]$: $i, j \in V$.

  The vertex set $V_1=\{1,2, …, k\}$, $V_2=\{k+2, …, n\}$, $V=V_1 \cup \{k+1\} \cup V_2$,

  $M_{k+1}[i,j]=1 \Leftrightarrow$ There exists a path i *to* j .

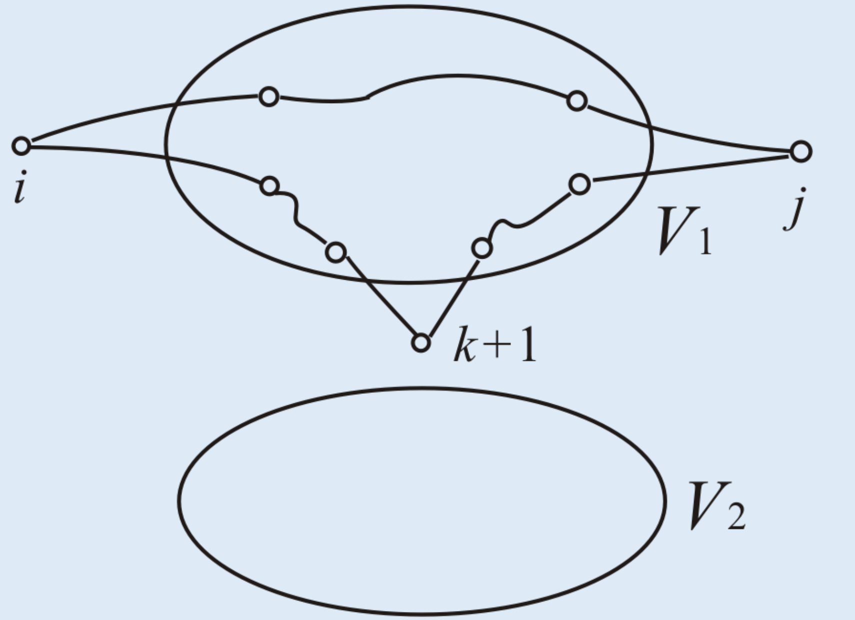
  that only passes through the points in $V_1 \cup \{k+1\}$.

- These paths are divided into two categories:
  - **Category 1:** Paths that only pass through the points in V_1
  - **Category 2:** Paths that pass through point $k+1$

For **Category 1** paths： $M_k[i,j]=1$

For **Category 2** paths:

$$M_k[i,k+1]=1 \wedge M_k[k+1,j]=1$$



Algorithm 4.1: Warshall Algorithm

Input: M （relation matrix of R ）

Output: M_t （relation matrix of $t(R)$）

1. $M_t \leftarrow M$
2. for $k \leftarrow 1$ to n do
3. for $i \leftarrow 1$ to n do
4. for $j \leftarrow 1$ to n do
5. $M_t[i, j] \leftarrow M_t[i, j]$ or $M_t[i, k] \cdot M_t[k, j]$

Time Complexity: $T(n)=O(n^3)$

Objective :

Key Concepts ：

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